e content for students of patliputra university

B. Sc. (Honrs) Part 2paper 4

Subject:Mathematics

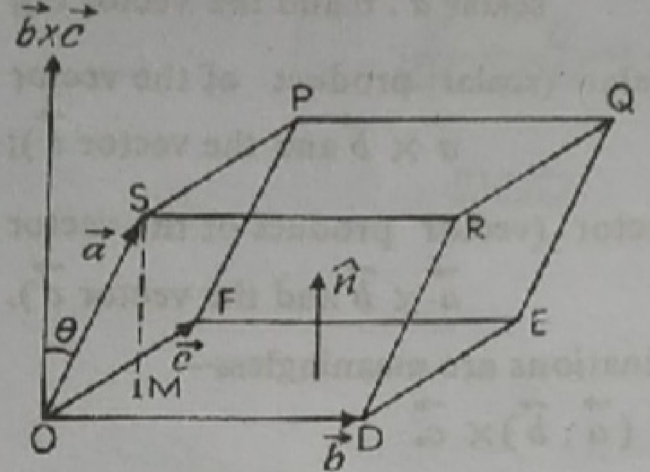
Topic:Products of three vectors

Scalar triple pruduct

Definition. If $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors, the dot product of one vector with the vector product of other two vectors i.e. the products of the type $\vec{a} \cdot (\vec{b} \times \vec{c})$, $(\vec{a} \times \vec{b}) \cdot \vec{c}$, $(\vec{a} \times \vec{c}) \cdot \vec{b}$ etc. are called *scalar triple products* of $\vec{a}, \vec{b}$ and $\vec{c}$.

Geometrical Interpretation of Scalar Triple Pr

To prove that $\vec{a}.(\vec{b} \times \vec{c})$ is numerically equal to the volume of a parallelopiped with $\vec{a}$, $\vec{b}$ and $\vec{c}$ as coterminous edges.



Proof. Let us consider a parallelopiped *ODEF PQRS* with coterminous edges $\vec{OS}=\vec{a}$, $\vec{OD}=\vec{b}$ and $\vec{OF}=\vec{c}$.

Let $\hat{n}$ be a unit normal to the parallelogram *ODEF* having the direction of $\vec{b}\times\vec{c}$. From *S* draw *SM* perpendicular to the face *ODEF*.

Let $SM=h$, and θ be the angle between the directions of $\hat{n}$ and $\vec{a}$.

Then $\vec{a}.(\vec{b}\times\vec{c})=\vec{a}\,(\text{area } ODEF)\,\hat{n}$

$=(\text{area } ODEF)\,\vec{a}\,.\,\hat{n}.$

$=(\text{area } ODEF).OS.1.\cos\theta$

$=(\text{area } ODEF).(\pm h),$

the sign + or − is taken according as θ is acute or obtuse, that is, according as $\vec{a}$, $\vec{b}$ and $\vec{c}$ form a right-handed system or a left-handed system

$=\pm V$, where V is the measure of the volume of the parallelopiped.

$\therefore\ \vec{a}.(\vec{b}\times\vec{c})=$ the volume of the parallelopiped (numerically).

Properties of Scalar Triple Product

(i) *In a scalar triple product the position of dot and cross can be interchanged*

i.e. $$\vec{a}.(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b}).\vec{c}.$$

Proof. We know that

$\vec{a}.(\vec{b}\times\vec{c})=$ the volume V of the parallelopiped with $\vec{a},\vec{b},\vec{c}$ as vectors represented by the coterminus edges and $\vec{a},\ \vec{b},\ \vec{c}$ forming a right-handed triad.

Similarly $\vec{c}.(\vec{a}\times\vec{b})=V$ and $\vec{b}.(\vec{c}\times\vec{a})=V.$

But $\vec{c}.(\vec{a}\times\vec{b})=(\vec{a}\times\vec{b}).\vec{c},\quad \{\because\ \vec{m}.\vec{n}=\vec{n}.\vec{m}\}$

$\therefore\quad (\vec{a}\times\vec{b}).\vec{c}=V.$

Hence $$\vec{a}.(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b}).\vec{c}.$$

$\therefore$ In a scalar triple product dot and cross can be interchanged.

Note. As the position of dot and cross is immaterial, both $\vec{a}.(\vec{b}\times\vec{c})$ and $(\vec{a}\times\vec{b}).\vec{c}$ are represented by

$$[\vec{a}\ \vec{b}\ \vec{c}] \text{ or } [\vec{a},\vec{b},\vec{c}].$$

So $$\boxed{\vec{a}.(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b}).\vec{c}=[\vec{a}\ \vec{b}\ \vec{c}].}$$

(ii) *In the scalar triple product* $[\vec{a}\ \vec{b}\ \vec{c}]$ *if* $\vec{a},\ \vec{b},\ \vec{c}$ *are changed in cyclic order the product does not change but if any two are interchanged the product changes its value in sign only.*

Proof. As above we can prove that

$$\vec{a}.(\vec{b}\times\vec{c})=\vec{b}.(\vec{c}\times\vec{a})=\vec{c}.(\vec{a}\times\vec{b}).$$

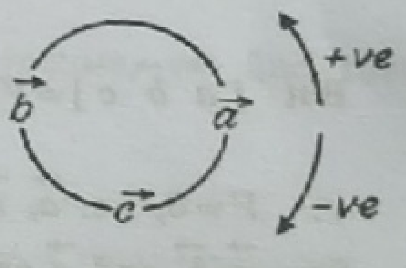
$\therefore\quad [\vec{a}\ \vec{b}\ \vec{c}]=[\vec{b}\ \vec{c}\ \vec{a}]=[\vec{c}\ \vec{a}\ \vec{b}].$

Now $\vec{b}.(\vec{a}\times\vec{c})=-\vec{b}.(\vec{c}\times\vec{a}),$

$\{\because\ \vec{a}\times\vec{c}=-\vec{c}\times\vec{a}\}$

$\therefore\quad [\vec{b}\ \vec{a}\ \vec{c}]=-[\vec{b}\ \vec{c}\ \vec{a}],$

$\therefore\quad [\vec{b}\ \vec{a}\ \vec{c}]=-[\vec{a}\ \vec{b}\ \vec{c}].$



Similarly,
$$[\vec{c}\ \vec{b}\ \vec{a}] = -[\vec{b}\ \vec{c}\ \vec{a}]$$
$$[\vec{a}\ \vec{c}\ \vec{b}] = -[\vec{c}\ \vec{a}\ \vec{b}].$$

Note. The above result can also be proved by using the method of evaluation in 2.6.

(iii) *The scalar triple product of three vectors vanishes if at least two of the vectors are equal.*

Proof.
$$[\vec{a}\ \vec{a}\ \vec{b}] = (\vec{a} \times \vec{a}).\vec{b} = \vec{0}.\vec{b} = 0;$$
$$[\vec{a}\ \vec{b}\ \vec{a}] = (\vec{a} \times \vec{b}).\vec{a} = \vec{a}.(\vec{a} \times \vec{b})$$
$$= (\vec{a} \times \vec{a}).\vec{b} = \vec{0}.\vec{b} = 0;$$
$$[\vec{a}\ \vec{a}\ \vec{a}] = (\vec{a} \times \vec{a}).\vec{a} = \vec{0}.\vec{a} = 0.$$

vector triple product

Definition. The *vector triple product* is the vector product of one with the vector product of the other two. If $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors, the products of the type $\vec{a} \times (\vec{b} \times \vec{c})$ and $(\vec{a} \times \vec{b}) \times \vec{c}$ are vector triple products.

By definition of vector product of two vectors, we find that $\vec{b} \times \vec{c}$ is a vector quantity.

Let $\vec{b} \times \vec{c} = \vec{d}$.

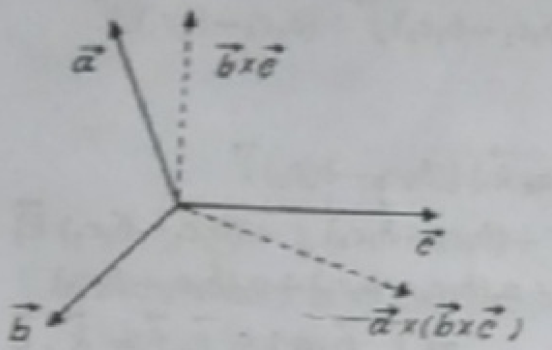
Then $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{a} \times \vec{d}$, which is again a vector quantity.

Therefore $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector quantity. But it is the product of three vectors. Hence $\vec{a} \times (\vec{b} \times \vec{c})$ is called vector product of three vectors.

To Find the Expansion of $\vec{a} \times (\vec{b} \times \vec{c})$ or, to prove that

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c},$$

Proof. We know that $\vec{b} \times \vec{c}$ is a vector perpendicular to the plane containing $\vec{b}$ and $\vec{c}$.

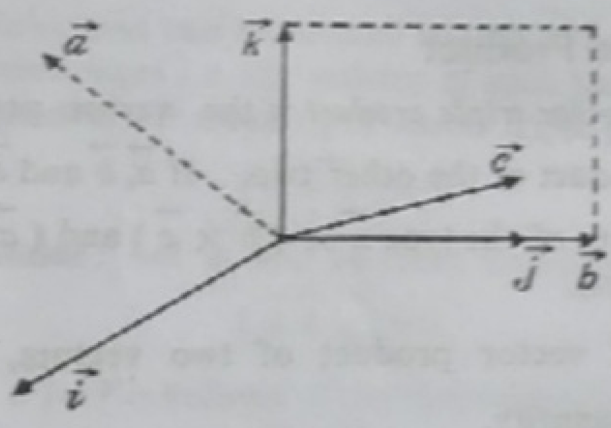


Likewise $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector perpendicular to the plane containing $\vec{a}$ and $\vec{b} \times \vec{c}$. Hence $\vec{a} \times (\vec{b} \times \vec{c})$ must lie in the plane of $\vec{b}$ and $\vec{c}$ and is perpendicular to $\vec{a}$.

Let us consider the unit vectors $\vec{i}, \vec{j}$ *and* $\vec{k}$.

Let $\vec{b}$ be along $\vec{j}$.

Therefore, let $\vec{b} = b_2 \vec{j}$.



Let $\vec{k}$ be perpendicular to $\vec{b}$ and in the plane of $\vec{b}$ and $\vec{c}$.

Therefore, $\vec{c}$ is in the plane of $\vec{j}$ and $\vec{k}$.

Let $\vec{c} = c_2\vec{j} + c_3\vec{k}$.

Let $\vec{a}$ be any arbitrary vector. Then we can take

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}.$$

Now $\vec{b} \times \vec{c} = b_2\vec{j} \times (c_2\vec{j} + c_3\vec{k})$

$= b_2c_3\vec{j} \times \vec{k} = b_2c_3\vec{i}$, as $\vec{j} \times \vec{j} = \vec{0}$, $\vec{j} \times \vec{k} = \vec{i}$.

Therefore $\vec{a} \times (\vec{b} \times \vec{c}) = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times b_2c_3\vec{i}$

$= a_2b_2c_3\vec{j} \times \vec{i} + a_3b_2c_3\vec{k} \times \vec{i}$, as $\vec{i} \times \vec{i} = \vec{0}$

$= -a_2b_2c_3\vec{k} + a_3b_2c_3\vec{j}$, .. (1)

as $\vec{j} \times \vec{i} = -\vec{k}$, $\vec{k} \times \vec{i} = \vec{j}$.

Again $(\vec{a} . \vec{c})\vec{b} = [(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) . (c_2\vec{j} + c_3\vec{k})]b_2\vec{j}$

$= (a_2c_2 + a_3c_3)b_2\vec{j}$,

as $\vec{i} . \vec{j} = 0, \vec{j} . \vec{j} = 1, \vec{k} . \vec{k} = 1$,

and $(\vec{a} . \vec{b})\vec{c} = [(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) . b_2\vec{j}](c_2\vec{j} + c_3\vec{k})$

$= a_2b_2(c_2\vec{j} + c_3\vec{k})$.

Therefore $(\vec{a} . \vec{c})\vec{b} - (\vec{a} . \vec{b})\vec{c}$

$= (a_2c_2 + a_3c_3)b_2\vec{j} - a_2b_2(c_2\vec{j} + c_3\vec{k})$

$= a_3c_3b_2\vec{j} - a_2b_2c_3\vec{k}$. .. (2)

From (1) and (2), we conclude that

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} . \vec{c})\vec{b} - (\vec{a} . \vec{b})\vec{c}.$$